COMPARISON TEST

(B.Sc.-II, Paper-III)

Group- B

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Comparison test

THEOREM :→ A series of non-negative term is convergent iff the sequence of its partial sum is bounded.

Proof :→ Let the series be

$$\sum u_n = u_1 + u_2 + \cdots\cdots + u_n + \cdots\cdots$$

Since each term of the series is non-negative,

$\therefore S_1 < S_2 < S_3 < \cdots\cdots$

$\therefore \{S_n\}$ is a monotonic increasing sequence.

$\therefore$ If $\{S_n\}$ is bounded then $\{S_n\}$ is convergent, and so $\sum u_n$ is convergent.

If $\sum u_n$ is ~~conv~~ convergent then, the sequence $\{S_n\}$ is convergent and so it is bounded.

proved.

Remark :→ A series of non-negative term is either convergent or divergent.

Comparison tests

THEOREM (Comparision test, First form):→

Let $\sum_{n\geq 1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} v_n$ be two series of non-negative terms and $K > 0$ is a constant such that

$$u_n \leq K v_n, \text{ for all } n \geq p, \text{ then}$$

(i) $\sum u_n$ is convergent if $\sum v_n$ is convergent.

(ii) $\sum v_n$ is divergent if $\sum u_n$ is divergent.

Proof:→

Let $S_n = u_1 + u_2 + \cdots\cdots + u_n$

and $t_n = v_1 + v_2 + \cdots\cdots + v_n$

Since convergence or divergence of a series is not affected by deleting a finite number of terms from the series.

Hence without loss of generality we can consider,

$$u_n \leq K v_n, \text{ for all } n \geq 1.$$

$$\therefore \quad u_1 + u_2 + \cdots + u_n \leq K(v_1 + \cdots + v_n)$$

Therefor, $S_n \leq K t_n, \ \forall n$ —— (1)

(i) If $\sum v_n$ is convergent then the sequence $\{t_n\}$ is convergent.

$\therefore \{t_n\}$ is bounded, and so there is a constant M s.t.

$$t_n \leq M, \ \forall n.$$

$\therefore$ $S_n \leq KM$ for all n.

$\therefore$ $\{S_n\}$ is bounded, and so $\{S_n\}$ is convergent.

$\therefore \sum u_n$ is convergent.

(ii) If $\sum u_n$ is divergent then the sequence $\{S_n\}$ is divergent.

$\therefore$ $\{S_n\}$ is unbounded.

$\therefore$ $\exists$ a real number M (however large) s.t.

$$S_n > (M \cdot K), \ \forall \ n \geq N \ (\text{for some } N \in \mathbb{N})$$

As $\{S_n\}$ is monotonic increasing sequence.

Now,

$$S_n \leq K t_n$$

$$\therefore \ M \cdot K < S_n \leq K t_n \ ; \ \forall \ n \geq N$$

$$\therefore \ t_n > M, \ \forall \ n \geq N$$

$\therefore$ $\{t_n\}$ is divergent and so $\sum v_n$ is divergent.

proved.

THEOREM (Comparision test, second form):→

If $\sum a_n$ and $\sum b_n$ are two series of Positive terms and L and K are fixed positive numbers such that

$$L b_n \leq a_n \leq K b_n, \text{ for all } n \geq P,$$

p being a fixed positive integer, then the series $\sum a_n$ and $\sum b_n$ converge or diverge together.

Proof:→

Let $S_n = a_1 + a_2 + \cdots + a_n$ and

$t_n = b_1 + b_2 + \cdots + b_n$ for $n \geq p$.

From the given condition

$$L \cdot (b_{p+1} + b_{p+2} + \cdots + b_n) \leq a_{p+1} + a_{p+2} + \cdots + a_n$$
$$\leq K (b_{p+1} + b_{p+2} \cdots + b_n)$$

i,e; $L \cdot (t_n - t_p) \leq S_n - S_p \leq K \cdot (t_n - t_p)$

Hence,

$$L t_n + (S_p - L \cdot t_p) \leq S_n \leq K \cdot t_n + (S_p - K \cdot t_p)$$

put $S_p - L \cdot t_p = A$, ~~Sp~~ and $S_p - K \cdot t_p = B$, then A and B are fixed real numbers.

Hence

$$L t_n + A \leq S_n \leq K \cdot t_n + B, \text{ for all } n \geq p. \quad -(1)$$

From ① it is clear that $\{S_n\}$ is bounded iff $\{t_n\}$ is bounded.

$\therefore \Sigma a_n$ is convergent iff Σb_n is convergent.

proved.

THEOREM (Comparison test, Limit form) :→

Let $\sum_{1}^{\infty} u_n$ and $\sum_{1}^{\infty} v_n$ be two series of positive terms, such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l \text{ (finite } \neq 0\text{), then}$$

(i) Σu_n is convergent iff Σv_n is convergent.

(ii) Σu_n is divergent iff Σv_n is divergent.

Proof:→

Since $u_n > 0$, $v_n > 0$, for all n.

$$\therefore \frac{u_n}{v_n} > 0, \text{ for all } n.$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} \geq 0 \Rightarrow l \geq 0$$

But $l \neq 0$ (hypothesis)

$\therefore l > 0$.

$\therefore$ We can have $\epsilon > 0$ s.t.

$l - \epsilon > 0$.

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

$\therefore$ There exists $N \in \mathbb{N}$ s.t.

$$0 < l - \epsilon < \frac{u_n}{v_n} < l + \epsilon, \text{ for all } n \geq N \quad \text{(B)}$$

~~and~~ $\therefore$ $$\frac{u_n}{v_n} < l + \epsilon, \; \forall n \geq N \quad \text{— (1)}$$

and $$\frac{u_n}{v_n} > l - \epsilon, \; \forall n \geq N$$

$$\therefore \quad u_n < (l+\epsilon) v_n, \; \forall n \geq N \quad \text{— (2)}$$

$$\& \quad u_n > (l-\epsilon) v_n, \; \forall n \geq N \quad \text{— (3)}$$

$\therefore$ By comparision test (first form)

If Σv_n is convergent then Σu_n is convergent by (2).

And if Σu_n is convergent then Σv_n is convergent by (3).

Also if Σv_n is divergent then Σu_n is divergent by (3).

And if Σu_n is divergent then Σv_n is divergent by (2).

$\therefore$ Σu_n and Σv_n converge or diverge together.

Proved.

THEOREM (Harmonic series):→

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent if $p>1$ and divergent if $p \leq 1$.

Proof:→

Case-1: When $p > 1$.

In the series of positive term, we can group the terms in brackets without ~~loss~~ ~~of~~ any change in the nature of series.

Let the term be bracketed as follows.

$$\sum \frac{1}{n^p} = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$+ \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \left(\frac{1}{16^p} + \cdots + \frac{1}{31^p}\right) + \cdots \quad -(A)$$

$$\because \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= \frac{4}{4^p} = \frac{2^2}{2^{2p}} = \frac{1}{2^{2(p-1)}}.$$

Similarly,

$$\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} < \frac{8}{8^p} = \frac{2^3}{2^{3p}} = \frac{1}{2^{3(p-1)}},$$

And so on.

$\therefore$ From (A)

$$\sum \frac{1}{n^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \text{------- to } \infty \quad \text{— (B)}$$

Right hand side of (B) is a G.P. series with

$$c.r = \frac{1}{2^{p-1}} < 1 \quad \left(\because p>1 \Rightarrow p-1>0 \right.$$
$$\Rightarrow 2^{p-1} > 2^0 = 1$$
$$\left. \Rightarrow \frac{1}{2^{p-1}} < 1 \right)$$

$\therefore$ Right hand side of (B) is convergent.

$\therefore$ By comparision test $\sum \frac{1}{n^p}$ is convergent.

Case-2 :→ When $p=1$.

We group the terms of the series in brackets as follows.

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \text{---} \right.$$
$$\left. \text{----} + \frac{1}{16}\right) + \text{--------} \quad \text{..} \quad \text{— (C)}$$

$$\because \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

And so on.

The sum of the series (C) after the second term are greater than the ~~series~~ corresponding term of the series.

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$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots\cdots \infty \quad \text{——— (D)}$$

In the series (D), $S_n = 1 + \frac{(n-1)}{2} \to \infty$ as $n \to \infty$.

$\therefore$ (D) is divergent

$\therefore$ By comparision test

series (C) is divergent.

<u>case-3</u>:→ If $p < 1$. then $n^p < n^1$

$$\therefore \frac{1}{n^p} > \frac{1}{n}$$

But $\sum \frac{1}{n}$ is divergent.

$\therefore \sum \frac{1}{n^p}$ is divergent.

<u>proved</u>.

Comparison test

Example ①:→ Test ~~the~~ for ~~the~~ convergence the series, whose nth term is.

$$\sqrt{n^4+1} - n^2$$

Solution:→ Here $a_n = \sqrt{n^4+1} - n^2$

$$\therefore a_n = \frac{\sqrt{n^4+1} - n^2}{\sqrt{n^4+1} + n^2} \times \frac{\sqrt{n^4+1} + n^2}{1}$$

$$= \frac{n^4+1 - n^4}{\sqrt{n^4+1} + n^2} = \frac{1}{\sqrt{n^4+1} + n^2}$$

Let $b_n = \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^4+1} + n^2} \times \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^4}} + 1}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2} \ (\neq 0 \text{ and finite})$$

$\therefore$ By comparision test Σa_n and Σb_n converge or diverge together.

But since $\Sigma b_n = \Sigma \frac{1}{n^2}$ is convergent.

Hence the series Σa_n is convergent.

Example ③ Test convergence of the series

$$1 + \frac{3}{5} + \frac{5}{13} + \frac{7}{25} + \cdots + \frac{2n-1}{2n^2-2n+1} + \cdots$$

Solution :→

Here $u_n = \dfrac{2n-1}{2n^2-2n+1}$

Let $v_n = \dfrac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n-1}{2n^2-2n+1} \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{2-\frac{1}{n}}{2-\frac{2}{n}+\frac{1}{n^2}}$$

$$= \frac{2}{2} = 1 \ (\neq 0 \ \& \text{ finite})$$

$\therefore$ By comparision test Σu_n and Σv_n are of same nature.

But since $\Sigma u_n = \Sigma \frac{1}{n}$ is divergent.

$\therefore$ Σu_n is also <u>divergent</u>.

Example ③ Test convergence of the series.

$$\frac{1}{2\cdot3\cdot4} + \frac{1}{4\cdot5\cdot6} + \frac{1}{6\cdot7\cdot8} + \cdots\cdots$$

Solution :→

Here $u_n = \dfrac{1}{2n\cdot(2n+1)\cdot(2n+2)}$

Let $v_n = \frac{1}{n^3}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^3}{2n\cdot(2n+1)\cdot(2n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{2\cdot\left(2+\frac{1}{n}\right)\cdot\left(2+\frac{2}{n}\right)}$$

$$= \frac{1}{8} \quad (\neq 0 \text{ \& finite})$$

$\therefore$ By comparision test, Σu_n and Σv_n are of same nature.

But since $\Sigma v_n = \Sigma \frac{1}{n^3}$ is convergent ($\because P=3>1$)

$\therefore$ Σu_n is convergent.

Example :→ Test convergence of the series

$$\Sigma\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$$

Solution :→

Here $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$= \frac{\left(\sqrt{n^4+1}\right)^2 - \left(\sqrt{n^4-1}\right)^2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$= \frac{n^4+1 - n^4+1}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$\therefore u_n = \frac{2}{\sqrt{n^4+1} - \sqrt{n^4-1}}$$

Let $v_n = \frac{1}{n^2}$

$$\therefore \frac{u_n}{v_n} = \frac{2n^2}{\sqrt{n^4+1} - \sqrt{n^4-1}}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n^2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}}$$

$$= \frac{2}{2} = 1 \quad (\neq 0 \text{ \& finite})$$

$\because$ By comarision test, Σu_n and Σv_n are of same nature.

But since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent.

$\therefore$ The series Σu_n is convergent.

Comparision test

Example :→ Test convergence of the series.

$$\sum \sin\frac{1}{n}.$$

Proof :→

Here $u_n = \sin\frac{1}{n} = \frac{1}{n} - \frac{1}{\lfloor 3 \cdot n^3} + \frac{1}{\lfloor 5 \cdot n^5} - \cdots\cdots \infty$

Let $v_n = \frac{1}{n}$, then

$$\therefore \frac{u_n}{v_n} = n\left(\frac{1}{n} - \frac{1}{\lfloor 3 \cdot n^3} + \cdots\cdots\right)$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = 1 \quad (\text{finite} \neq 0)$$

$\because \sum v_n = \sum \frac{1}{n}$ is divergent.

$\therefore \sum u_n$ is also divergent.

Thank you